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ASYMMETRIC MAXIMAL IDEALS IN $M(G)$

BY

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ABSTRACT. Let G be a nondiscrete LCA group, $M(G)$ the measure algebra of G, and $M_0(G)$ the closed ideal of those measures in $M(G)$ whose Fourier transforms vanish at infinity. Let Δ_G, Σ_G and Δ_0 be the spectrum of $M(G)$, the set of all symmetric elements of Δ_G, and the spectrum of $M_0(G)$, respectively. In this paper this is shown: Let Φ be a separable subset of $M(G)$. Then there exist a probability measure τ in $M_0(G)$ and a compact subset X of $\Delta_0 \backslash \Sigma_G$ such that for each $|c| \leqslant 1$ and each

$$\nu \in \Phi \ \operatorname{Card}\{f \in X: \hat{\tau}(f) = c \text{ and } |\hat{\nu}(f)| = r(\nu)\} \geqslant 2^{\mathfrak{c}}.$$

Here $r(\nu) = \sup\{|\hat{\nu}(f)|: f \in \Delta_G \backslash \hat{G}\}$. As immediate consequences of this result, we have (a) every boundary for $M_0(G)$ is a boundary for $M(G)$ (a result due to Brown and Moran), (b) $\Delta_G \backslash \Sigma_G$ is dense in $\Delta_G \backslash \hat{G}$, (c) the set of all peak points for $M(G)$ is $\hat{G}$ if G is σ-compact and is empty otherwise, and (d) for each $\mu \in M(G)$ the set $\hat{\mu}(\Delta_0 \backslash \Sigma_G)$ contains the topological boundary of $\hat{\mu}(\Delta_G \backslash \hat{G})$ in the complex plane.

Throughout the paper, let G be a nondiscrete locally compact abelian group with dual $\hat{G}$, $M(G)$ the convolution measure algebra of G, and $M_0(G)$ the ideal in $M(G)$ which consists of all measures with Fourier transforms vanishing at infinity. As is well known, we then have $L^1(G) = M_a(G) \subset M_0(G) \subset M_c(G)$. Let Δ_G denote the spectrum of $M(G)$, i.e., the space of all nonzero complex homomorphisms of $M(G)$, and let $\hat{\mu}$ denote the Gelfand transform of $\mu \in M(G)$. We define

$$\Delta_0 = \{f \in \Delta_G: \hat{\sigma}(f) \neq 0 \text{ for some } \sigma \in M_0(G)\},$$

$$\Sigma_G = \{f \in \Delta_G: f(\sigma^*) = \overline{f(\sigma)} \text{ for all } \sigma \in M(G)\},$$

where $\sigma^*(E) = \overline{\sigma(-E)}$ for all Borel sets E in G and $f(\sigma) = \hat{\sigma}(f)$. Then Δ_0 is open (in Δ_G), Σ_G is closed, and $\hat{G} \subset \Delta_0 \cap \Sigma_G$. Moreover, Δ_0 may be identified with the spectrum of $M_0(G)$, since $M_0(G)$ is an ideal in $M(G)$.

It is shown in [11] that given $\mu \in M_c(G)$, there exist fairly many elements $f \in \Delta_G$ such that $M_a(G) + L^1(\mu) \subset \operatorname{Ker}(f)$ but $M_0(G) \not\subset \operatorname{Ker}(f)$. In fact, it is not difficult to improve Theorem 2 of [11] as follows.

THEOREM A. *Let $0 \neq \lambda \in M_0(G)$, $\mu \in M_c(G)$, and H a subgroup of G which is a G_δ-set. Then there exists a probability measure $\sigma = \tau * \tau^*$, with $\tau \in$*

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$M_0^+(\operatorname{supp}\lambda)$, *having the following properties*:

(i) *Given* $0 \leqslant r \leqslant 1$, *the set of all* $f \in \Sigma_G$ *such that*

$$\hat{\sigma}(f) = r, \quad \operatorname{Ker}(f) \supset L^1(\mu), \quad \textit{and} \quad \hat{\nu}(f) = \hat{\nu}(1) \quad \forall\, \nu \in M_d(G)$$

has cardinality $\geqslant 2^{\mathfrak{c}}$. *Here* $\mathfrak{c}$ *denotes the cardinal number of the continuum.*

(ii) *Given a complex number* c *of modulus* $\leqslant 1$ *and* $g \in \Delta_G$ *with* $g(\delta_x) = 1$ *for all* $x \in H$, *the set of all* $f \in \Delta_G \backslash \Sigma_G$ *such that*

$$\hat{\sigma}(f) = c, \quad \operatorname{Ker}(f) \supset L^1(\mu), \quad \textit{and} \quad \hat{\nu}(f) = \hat{\nu}(g) \quad \forall\, \nu \in M_d(G)$$

has cardinality $\geqslant 2^{\mathfrak{c}}$.

For some related results, we refer the reader to Izuchi and Shimizu [8], Saka [12], Shimizu [13], and Williamson [15]. Now let $\mu \in M(G)$ be given, and define

$$r(\mu) = \sup\{|\hat{\mu}(f)| : f \in \Delta_G \backslash \hat{G}\}.$$

Since $\Delta_G \backslash \hat{G}$ is compact, there exists at least one f in this set such that $|\hat{\mu}(f)| = r(\mu)$. It seems to be a natural problem to ask how many f as above there exist. Our answer is as follows.

THEOREM B. *Let* $\mu \in M(G)$, *and* Φ *a separable subset of* $L^1(\mu)$. *Then there exist a probability measure* τ *in* $M_0(G)$ *and a compact set* X *in* $\Delta_0 \backslash \Sigma_G$ *such that*

$$\operatorname{Card}\{f \in X : \hat{\tau}(f) = c \textit{ and } |\hat{\nu}(f)| = r(\nu)\} \geqslant 2^{\mathfrak{c}}$$

for every complex number c *of modulus* $\leqslant 1$ *and every measure* ν *in* $[L^1(\mu) \cap M^+(G)] \cup \Phi$.

Notice that we can set $\Phi = L^1(\mu)$ if G is metrizable, since then $L^1(\mu)$ is separable. As easy consequences of the last theorem, we have the following results.

COROLLARY 1.

(a) *Every boundary of* $M_0(G)$ *is a boundary of* $M(G)$.

(b) *The set* $\Sigma_G \backslash \hat{G}$ *is the topological boundary of* $\Delta_G \backslash \Sigma_G$ *in* Δ_G. *In other words,* $\Delta_G \backslash \Sigma_G$ *is dense in* $\Delta_G \backslash \hat{G}$.

(c) *If* G *is* σ-*compact, then the set* P_G *of all peak points for* $M(G)$ *is precisely* $\hat{G}$. *If not, then* $P_G = \emptyset$.

COROLLARY 2. *For each* $\mu \in M(G)$, *the set* $\hat{\mu}(\Delta_0 \backslash \Sigma_G)$ *contains the topological boundary of* $\hat{\mu}(\Delta_G \backslash \hat{G})$ *in the complex plane* $\mathbf{C}$. *In particular, we have*

(a) *If* $\operatorname{Card}[\hat{\mu}(\Delta_0 \backslash \Sigma_G)] < \mathfrak{c}$, *then* $\hat{\mu}(\Delta_G \backslash \hat{G})$ *is (at most) countable and coincides with* $\hat{\mu}(\Delta_0 \backslash \Sigma_G)$.

(b) *If* $\hat{\mu}$ *is real on* $\Delta_0 \backslash \Sigma_G$, *then* $\hat{\mu}(\Delta_G \backslash \hat{G}) = \hat{\mu}(\Delta_0 \backslash \Sigma_G)$.

Notice that Theorem B implies the result of Brown and Moran [2] and Graham [5]: If $\mu \in M(G)$ and $\hat{\mu} = 0$ off Σ_G, then $r(\mu) = 0$. Part (a) of Corollary 1 is due to Brown and Moran [3]. We also refer to Brown's result in [1]: $\Delta_0 \cap \Sigma_G$ is not entirely contained in the Shilov boundary of $M_0(G)$. It may be an interesting problem to ask whether or not we have $\hat{\mu}(\Delta_0 \backslash \Sigma_G) = \hat{\mu}(\Delta_G \backslash \hat{G})$ for all $\mu \in M(G)$.

To prove Theorem B, we shall first construct a measure of a certain type (assuming that G is metrizable). The construction of such a measure is almost the same as the corresponding one in [11], and Körner's method [9] plays an important role in our construction.

We now introduce some notation. Let m_G denote the Haar measure of G, and $\mathbf{Z}$ the group of all integers. For a set K in G and $p \in \mathbf{Z}^+$, we define

$$pK = \{x_1 + \cdots + x_p : x_j \in K \text{ for all } 1 \leqslant j \leqslant p\}$$

if $p \geqslant 1$, $pK = \{0\}$ if $p = 0$, and $(-p)K = -(pK)$. The subgroup of G which the set K generates is denoted by $Gp(K)$. We say that a Borel set K is of type M_0 if $M_0(K) = M_0(G) \cap M(K)$ is nonzero. Let $q(G)$ denote the supremum of all natural numbers p such that every neighborhood of the identity 0 of G contains an element of order $\geqslant p$. Then it is easy to see that if $q(G) = \infty$, then G is an I-group, and that if $q(G)$ is finite, then G contains an open-and-compact subgroup H such that $\operatorname{ord}(x) \leqslant q(G)$ for all x in H. A set K in G is called *strongly independent* if it is independent in the usual sense [10, p. 97] and if all of its elements have order $q(G)$. Finally, we denote by $Gp'(K)$ the set of all points x of the form $x = k_1 x_1 + \cdots + k_u x_u$, where $u = u_x$ is a natural number, $x_1, \ldots, x_u$ are distinct elements of K, $k_j \in \mathbf{Z}$ for all $1 \leqslant j \leqslant u$, and $|k_j| = 1$ for at least one index j.

LEMMA 1. *Let $\mu_0 \in M^+(G)$, D a compact subset of G with Haar measure zero, and N a natural number. Let also $V_1, V_2, \ldots, V_u$ be nonempty open sets in G. Then we can find nonempty open sets $U_j \subset V_j$ $(1 \leqslant j \leqslant u)$ subject to the following conditions*:

(i) *If $p_j \in \mathbf{Z}$, $|p_j| < q(G)$, and $1 \leqslant \Sigma_{j=1}^u |p_j| \leqslant N$, then the set $\Sigma_{j=1}^u p_j U_j$ does not contain $0 \in G$, and*

$$m_G\left[D + \sum_{j=1}^{u} p_j U_j\right] < 1/N.$$

(ii) *If $q_j \in \mathbf{Z}$, $\Sigma_{j=1}^u |q_j| \leqslant N$, and $|q_j| = 1$ for at least one index j, then*

$$\mu_0\left[D + \sum_{j=1}^{u} q_j U_j\right] < 1/N.$$

PROOF. Let P be the set of all $p = (p_1, \ldots, p_u) \in \mathbf{Z}^u$ as in (i). Similarly, let Q be the set of all $q = (q_1, \ldots, q_u) \in \mathbf{Z}^u$ as in (ii).

The standard Baire category argument [10, 5.2.3] shows that there are points $x_j \in V_j$ $(1 \leqslant j \leqslant u)$ of order $\geqslant q(G)$ such that $\{x_j\colon 1 \leqslant j \leqslant u\}$ is independent. Since P is finite and D is a compact set with Haar measure zero, we can find a neighborhood W of $0 \in G$ so that

$$(1) \qquad 0 \notin \sum_1^u p_j(x_j + W) \quad \text{and} \quad m_G\left[D + \sum_1^u p_j(x_j + W)\right] < 1/N$$

for all $p \in P$. We may assume that $x_j + W \subset V_j$ $(1 \leqslant j \leqslant u)$.

Put $E = \{x_j\colon 1 \leqslant j \leqslant u\}$, and take a compact neighborhood X of $0 \in G$ such that $X + X \subset W$. Since $M_a(G)$ is an ideal of $M(G)$, it follows from the Fubini theorem and the definition of Q that

$$(2) \qquad \int_{X^u} \sum_{q \in Q} \mu_0\left[D + Gp(E) + \sum_1^u q_j t_j\right] dt_1 \cdots dt_u = 0.$$

Therefore there are u points $t_1, t_2, \ldots, t_u$ in X for which the integrand in (2) is zero. Hence, in particular, we have

$$(3) \qquad \mu_0\left[D + \sum_1^u q_j y_j\right] = 0 \qquad (q \in Q),$$

where $y_j = x_j + t_j$. Upon comparing (1) and (3), we see that if $U \subset X$ is a sufficiently small neighborhood of $0 \in G$, then the sets $U_j = y_j + U$ have the required properties.

LEMMA 2. *Suppose that G is metrizable. Let $\mu_0 \in M^+(G)$, and let C_0 be a σ-compact subset of G with Haar measure zero. Then there exists a strongly independent compact set K in G of type M_0 such that*

$$m_G[C_0 + Gp(K)] = \mu_0[C_0 + Gp'(K)] = 0.$$

PROOF. If $q(G)$ is finite, we fix an open-and-compact subgroup H of G such that $\operatorname{ord}(x) \leqslant q(G)$ for all x in H. In the other case, we set $H = G$.

Let $\{D_n\}_1^\infty$ be an increasing sequence of compact subsets of G with $C_0 = \bigcup_1^\infty D_n$, and $\{\hat{E}_n\}_1^\infty$ a sequence of compact subsets of $\hat{G}$ with $\hat{G} = \bigcup_1^\infty \hat{E}_n$. We shall construct a sequence $\{\sigma_n\}_1^\infty$ of probability measures in $M_0(H)$, a sequence $\{\mathcal{I}_n\}_1^\infty$ of finite collections of disjoint compact sets in H, a sequence $\{\hat{F}_n\}_1^\infty$ of compact sets in $\hat{G}$, and also a sequence $\{n_p\}_1^\infty$ of natural numbers. They will satisfy the following conditions (and some other conditions):

$$(1) \qquad \operatorname{supp} \sigma_n \subset \bigcup \{\operatorname{int}(I)\colon I \in \mathcal{I}_n\}.$$

$$\sup\{|\widehat{|\sigma_n|_I}(\chi)|: \chi \in \hat{G}\backslash\hat{F}_n\} < 2^{-n}\sigma_n(I) \quad \forall I \in \mathcal{I}_n. \tag{2}$$

It is also assumed that each set in $\mathcal{I}_{n+1}$ is a subset of some set in $\mathcal{I}_n$.

We first take any probability measure $\sigma_1 \in M_0(H)$ with compact support of diameter $< 1/2$. Let I be any compact neighborhood of supp σ_1 such that diam $I < 1$, $\mathcal{I}_1 = \{I\}$, and $n_1 = 1$. Since $\sigma_1 \in M_0(G)$, we can take a compact set $\hat{F}_1$ in $\hat{G}$ subject to (2) with $n = 1$.

Suppose that p is a natural number, and that n_j $(1 \leqslant j \leqslant p)$, σ_n, $\mathcal{I}_n$, $\hat{F}_n$ $(1 \leqslant n \leqslant m = n_p)$ have been defined. Let M_p be the largest natural number such that

$$\max\{\sigma_m(I): I \in \mathcal{I}_m\} \leqslant M_p^{-2}, \tag{3}$$

and write

$$\{A \subset \mathcal{I}_m: 1 \leqslant \text{Card } A \leqslant M_p\} = \{A_r: 1 \leqslant r \leqslant s_p\}. \tag{4}$$

Setting $n_{p+1} = n_p + s_p$, we shall construct $\sigma_n, \mathcal{I}_n$, and $\hat{F}_n$ for all $m < n \leqslant n_{p+1}$ as follows.

Suppose that these objects have been defined for some $n = m + r - 1$ with $1 \leqslant r \leqslant s_p$, and put

$$\mathcal{K}_n = \{I \in \mathcal{I}_n: I \subset J \text{ for some } J \in A_r\}. \tag{5}$$

Then, for each set K in $\mathcal{K}_n$, there are a finite collection $\{L_j^K\}_j$ of disjoint compact sets in K and a collection $\{\nu_j^K\}_j$ of measures in $M_0^+(K)$, with supp $\nu_j^K \subset$ int(L_j^K), such that

$$0 < \|\nu_j^K\| < n^{-1}\sigma_n(K); \tag{6}$$

$$\sum_j \|\nu_j^K\| = \sigma_n(K); \tag{7}$$

$$\left|\sum_j (\nu_j^K)\hat{}\,(\chi) - (\sigma_n|K)\hat{}\,(\chi)\right| < 2^{-n}\sigma_n(K) \quad \forall \chi \in \hat{F}_n. \tag{8}$$

To see this, it suffices to apply Lemma 3 of [11] and its obvious modification. By virtue of Lemma 1, we can demand that the sets L_j^K satisfy the following additional conditions:

$$\text{diam } L_j^K < 1/n; \tag{9}$$

$$0 \notin \sum_{K \in \mathcal{K}_n} \sum_j p_j^K L_j^K \quad \forall (p_j^K) \in P_n; \tag{10}$$

$$(11)\qquad m_G\Big[D_n + \sum_{K\in \mathcal{K}_n}\sum_j p_j^K L_j^K\Big] < 2^{-n}/\mathrm{Card}\, P_n \quad \forall (p_j^K)\in P_n;$$

$$(12)\qquad \mu_0\Big[D_n + \sum_{K\in \mathcal{K}_n}\sum_j q_j^K L_j^K\Big] < 2^{-n}/\mathrm{Card}\, Q_n \quad \forall (q_j^K)\in Q_n.$$

Here P_n is the set of all tuples (p_j^K) of integers such that $|p_j^K| < q(G)$ for all j and K and $1 \leqslant \Sigma_{K,j}|p_j^K| \leqslant n$. Similarly Q_n is the set of all tuples (q_j^K) of integers such that $|q_j^K| = 1$ for some (K, j) and $\Sigma_{K,j}|q_j^K| \leqslant n$. Define

$$(13)\qquad \sigma_{n+1} = \sum_{I\notin \mathcal{K}_n} \sigma_n|I + \sum_{K\in\mathcal{K}_n}\sum_j \nu_j^K,$$

$$(14)\qquad \mathcal{I}_{n+1} = (\mathcal{I}_n\setminus \mathcal{K}_n)\cup \bigcup_{K\in\mathcal{K}_n}\{L_j^K\}_j.$$

Then (1), with n replaced by $n + 1$, is satisfied. Finally we choose a compact set $\hat{F}_{n+1}$ in $\hat{G}$, with $\hat{F}_{n+1} \supset \hat{F}_n \cup \hat{E}_n$, so that (2) holds for $n + 1$.

This completes our induction. It is a routine matter to prove that the sequence $\{\sigma_n\}_1^\infty$ converges to some probability measure $\sigma\in M_0(H)$ in the weak-* topology of $M(G)$, that

$$(15)\qquad K = \mathrm{supp}\,\sigma \subset \bigcap_{n=1}^{\infty}\bigcup\{I: I\in \mathcal{I}_n\},$$

and that K is strongly independent. (See the proof of Lemma 4 of [11], and notice that every element of H has order $\leqslant q(G)$.)

Now we want to confirm

$$m_G[C_0 + Gp(K)] = \mu_0[C_0 + Gp'(K)] = 0.$$

Let $0 \neq x \in Gp(K)$ be given. We have $x = \Sigma_1^u k_i x_i$ for some $(k_1, \ldots, k_u)\in \mathbf{Z}^u$ and some distinct elements $x_1, \ldots, x_u$ of K. By (9), (14), and (15), there exists a natural number N_x such that the points x_i belong to distinct sets in $\mathcal{I}_n$ whenever $n > N_x$. Choose any natural number p so that

$$n_p > N_x + \sum_1^u |k_i| \quad\text{and}\quad M_p > u,$$

and let $\mathcal{A}$ be the collection of all I in $\mathcal{I}_{n_p}$ which contain some x_i $(1 \leqslant i \leqslant u)$. Then $1 \leqslant \mathrm{Card}\,\mathcal{A} = u < M_p$, and so $\mathcal{A} = \mathcal{A}_r$ for some $1 \leqslant r \leqslant s_p$ by (4). Setting $n = n_p + r - 1$, we therefore infer from (5), (14) and (15) that x belongs to the set

$$\bigcup_{P_n}\Big(\sum_{K\in\mathcal{K}_n}\sum_j p_j^K L_j^K\Big).$$

Since p can be chosen as large as one pleases, we conclude that

$$Gp(K)\backslash\{0\} \subset \bigcup_{n=N}^{\infty} \bigcup_{P_n} \sum_{K_n} \sum_j p_j^K L_j^K \qquad (N = 1, 2, \ldots). \tag{16}$$

Similarly we have

$$Gp'(K) \subset \bigcup_{n=N}^{\infty} \bigcup_{Q_n} \sum_{K_n} \sum_j q_j^K L_j^K \qquad (N = 1, 2, \ldots). \tag{17}$$

It follows from (11) and (16) that

$$\begin{aligned} m_G[D_N + Gp(K)] &\leqslant \sum_{n=N}^{\infty} \sum_{P_n} m_G\left[D_N + \sum_{K_n} \sum_j p_j^K L_j^K\right] \\ &\leqslant \sum_{n=N}^{\infty} \sum_{P_n} m_G\left[D_n + \sum_{K_n} \sum_j p_j^K L_j^K\right] < 2^{-N+1} \end{aligned} \tag{18}$$

for all $N \geqslant 1$. (Notice that $m_G(D_N) = 0$.) Letting $N \longrightarrow \infty$ in (18), we have $m_G[C_0 + Gp(K)] = 0$. Similarly we have $\mu_0[C_0 + Gp'(K)] = 0$ by (12) and (17). This completes the proof.

LEMMA 3. *Let $\mu_0 \in M^+(G)$, C_0 a σ-compact subset of G which carries μ_0, and K a compact subset of G such that*

$$\mu_0[C_0 + Gp'(K)] = 0. \tag{$*$}$$

Suppose that $K_1, K_2, \ldots, K_p$ are disjoint compact subsets of K and that $\sigma_j \in M_c(K_j \cup (-K_j))$ for all $1 \leqslant j \leqslant p$.

(a) *If $m = (m_j)_1^p$ and $n = (n_j)_1^p$ are different tuples of nonnegative integers, then*

$$\mu_0 * \sigma_1^{m_1} * \cdots * \sigma_p^{m_p} \perp \mu_0 * \sigma_1^{n_1} * \cdots * \sigma_p^{n_p}.$$

(b) *If $\sigma_j \in M_c(K_j)$ for all $1 \leqslant j \leqslant p$ and $\nu \in L^1(\mu_0)$, then*

$$\|\nu * \sigma_1^{n_1} * \cdots * \sigma_p^{n_p}\| = \|\nu\| \cdot \|\sigma_1\|^{n_1} \cdots \|\sigma_p\|^{n_p}.$$

PROOF. To prove (a), we use the well-known method of Hewitt and Kakutani [6] (see also [10, 5.4.2]). Without loss of generality, assume that $\sigma_j \geqslant 0$ for all $1 \leqslant j \leqslant p$ and that $m_1 < n_1$. Write $\tau_m = \sigma_1^{m_1} * \cdots * \sigma_p^{m_p}$, and similarly for τ_n. Putting $E_j = K_j \cup (-K_j)$ for $1 \leqslant j \leqslant p$, we then see that $\mu_0 * \tau_m$ is carried by the set $A_m = C_0 + m_1 E_1 + \cdots + m_p E_p$. Therefore it suffices to show $(\mu_0 * \tau_n)(A_m) = 0$. Let $\lambda_j \in M(E_j^{n_j})$ be the n_j-fold product of σ_j, and let B_j be the set of all points $x_j = (x_{j1}, \ldots, x_{jn_j})$ of $E_j^{n_j}$ such that $x_{ji} \neq \pm x_{jk}$ whenever $1 \leqslant i < k \leqslant n_j$. Since σ_j is a continuous measure, we then have

$\lambda_j(G^{n_j}\backslash B_j) = 0$ by the Fubini theorem. On the other hand, $(x_{ji}) \in B_1 \times \cdots \times B_p$ implies

$$(1) \qquad \mu_0\left[A_m - \sum_{j=1}^{p}\sum_{i=1}^{n_i} x_{ji}\right] \leqslant \mu_0\left[C_0 + m_1E_1 - \sum_{i=1}^{n_1} x_{1i} + \sum_{j=2}^{p} Gp(K_j)\right]$$

$$\leqslant \mu_0[C_0 + Gp'(K)] = 0$$

by (*) and the definition of $Gp'(K)$. Evidently these two facts imply $(\mu_0 * \tau_n)(A_m) = 0$, as was required.

To prove (b), we need the following fact: Given $\mu \in M(G)$ and $\epsilon > 0$, there is a neighborhood V of $0 \in G$ such that

$$(2) \qquad \sigma \in M^+(G), \quad \operatorname{supp}\sigma - \operatorname{supp}\sigma \subset V \Rightarrow \|\mu * \sigma\| \geqslant (\|\mu\| - \epsilon)\|\sigma\|.$$

Suppose by way of contradiction that this is false for some μ and ϵ. Then, to each neighborhood V of 0 there corresponds a probability measure $\sigma_V \in M(G)$ such that $\|\mu * \sigma_V\| < \|\mu\| - \epsilon$ and $\operatorname{supp}\sigma_V \subset V - x_V$ for some $x_V \in G$. Upon replacing σ_V by $\sigma_V * \delta_{x_V}$, we may assume that $x_V = 0$. But then the net $\{\mu * \sigma_V\}$ converges to μ in the weak-* topology of $M(G)$. Hence

$$\|\mu\| \leqslant \liminf_V \|\mu * \sigma_V\| \leqslant \|\mu\| - \epsilon,$$

a contradiction.

We now prove (b) as follows. By the continuity of convolution, we can retain generality in assuming that each σ_j has the form $\sigma_j = \Sigma_{k=1}^q c_{jk}\tau_{jk}$, where the c_{jk} are complex numbers of absolute modulus one and the τ_{jk} are mutually singular measures in $M_c^+(K_j)$. Expanding $\sigma_j^{n_j} = (\Sigma_{k=1}^q c_{jk}\,\tau_{jk})^{n_j}$ for all $1 \leqslant j \leqslant p$ and applying part (a), we reduce (b) to the case where $\sigma_j \geqslant 0$ $(1 \leqslant j \leqslant p)$, and hence to the case where $c_{jk} = 1$ for all j and k. Since we can demand that every τ_{jk} has support of sufficiently small diameter, part (b) follows from (2). This completes the proof.

Proof of Theorem B. Let $\mu \in M(G)$, and Φ a separable subset of $L^1(\mu)$. Given $\sigma \in M(G)$, we let σ_s denote the singular part of σ with respect to m_G. Notice that

$$(*) \qquad r(\sigma) = \lim_{n\to\infty} \|\sigma^n + M_a(G)\|^{1/n} = \lim_{n\to\infty} \|(\sigma^n)_s\|^{1/n},$$

since $M_a(G)$ is an ideal in $M(G)$ with spectrum $\hat{G}$. Now define μ_0 to be the singular part of $\exp(|\mu|)$, and choose a σ-compact subset C_0 of G so that $m_G(C_0) = \mu_0(G\backslash C_0) = 0$. Then $\nu \in L^1(\mu)$ implies $(\nu^n)_s \in L^1(\mu_0)$ for all $n \in \mathbf{Z}^+$.

We first assume that G is metrizable, and take a compact subset K of G as where $H = H_\Gamma$ is the annihilator of Γ in G and m_H denotes the Haar measure of H of norm one. This can be proved by considering the Fourier transform of ν and by applying Theorem 1.9.1 of [10]. Since $\Phi \subset L^1(\mu)$ is separable, there is a σ-compact open subgroup Γ of G such that

$$(9) \qquad \|(\nu^n)_s\| = \|(\nu^n)_s * m_H\| \quad \forall \nu \in \Phi \text{ and } \forall n \in \mathbf{Z}^+.$$

By Lemma 6 of [11], we may assume that $G_0 = G/H$ is metrizable and $m_G(C_0 + H) = 0$. Let π: $G \longrightarrow G_0$ be the natural quotient map, and let

$$\nu \longrightarrow \pi^*(\nu) = \nu \circ \pi^{-1}: M(G) \longrightarrow M(G_0)$$

be the measure algebra homomorphism induced by π. Then it is easy to check that π^* maps $M_a^+(G)$ onto $M_a^+(G_0)$, $M_0^+(G)$ onto $M_0^+(G_0)$, and $L^1(\mu_0)$ onto $L^1(\pi^*(\mu_0))$ (cf. [14, 2.2.4]). Moreover, we have $\|\pi^*(\nu)\| = \|\nu * m_H\|$ for all $\nu \in M(G)$, as is easily seen. It follows from (9) that

$$(10) \qquad \|\pi^*[(\nu^n)_s]\| = \|(\nu^n)_s\| \quad \forall n \in \mathbf{Z}^+$$

for all $\nu \in \Phi$. Obviously (10) is satisfied for every $\nu \in M^+(G)$ as well.

Since $m_{G_0}[\pi(C_0)] = m_G(C_0 + H) = 0$ and $\pi^*(\mu_0)$ is carried by the set $\pi(C_0)$, we have $L^1(\pi^*(\mu_0)) \subset M_s(G_0)$. In particular $\pi^*[(\nu^n)_s]$ is the singular part of $(\pi^*(\nu))^n = \pi^*(\nu^n)$ for every $\nu \in L^1(\mu)$ and every $n \in \mathbf{Z}^+$. Hence $r[\pi^*(\nu)] = r(\nu)$ for all $\nu \in [L^1(\mu) \cap M^+(G)] \cup \Phi$, by (10). To complete the proof, it therefore suffices to note that $\pi^*[M_0^+(G)] = M_0^+(G_0)$, that $\pi^*[M(G)] = M(G_0)$, and that the adjoint map of π^* sends $\Delta_{G_0}\backslash\Sigma_{G_0}$ into $\Delta_G\backslash\Sigma_G$ in a one-to-one way. This establishes Theorem B for all nondiscrete groups.

Proof of Corollary 1. Let $Y \subset \Delta_0$ be a boundary of $M_0(G)$, and $\mu \in M(G)$. Choose any $f \in \Delta_G$ such that $|\hat{\mu}(f)| = \|\hat{\mu}\|_{\Delta_G}$. If $f \in \hat{G}$, we take $\lambda \in M_a(G)$ so that $0 \leqslant \hat{\lambda} \leqslant 1$ on $\hat{G}$ and $\hat{\lambda}(f) = 1$. Then we have $\lambda * \mu \in M_0(G)$ and $\|\widehat{\lambda * \mu}\|_{\Delta_G} = |\hat{\mu}(f)|$; hence $|\hat{\mu}(g)| = |\widehat{\lambda * \mu}(g)| = |\hat{\mu}(f)|$ for some $g \in Y$. If $f \notin \hat{G}$, then $r(\mu) = |\hat{\mu}(f)|$. By Theorem B, we can find a probability measure $\tau \in M_0(G)$ such that $r(\tau * \mu) = r(\mu)$. Then $|\hat{\mu}(g)| = |\widehat{\tau * \mu}(g)| = r(\mu) = |\hat{\mu}(f)|$ for some $g \in Y$, which establishes part (a).

To prove (b), first notice that $\overline{\Delta_G\backslash\Sigma_G} \subset \Delta_G\backslash\hat{G}$ since $\hat{G}$ is open and is contained in Σ_G. If the above two sets were different, there would exist a nonempty open set U in Δ_G such that $U \cap \overline{\Delta_G\backslash\Sigma_G} = \varnothing \neq U\backslash\hat{G}$. Since the space of all Gelfand transforms of measures is closed under the complex conjugation on Σ_G, it would follow from the Stone-Weierstrass theorem that there would exist a $\hat{\mu} \in M(G)$ such that $0 \leqslant \hat{\mu} \leqslant 1$ on Σ_G, $\hat{\mu}(f) = 1$ for some $f \in U\backslash\hat{G}$, and $\hat{\mu} < 1/2$ on $\Sigma_G\backslash U$. Then the set $U \cap \hat{\mu}^{-1}(1)$ would be a local peak set for $M(G)$, and therefore would be a peak set for $M(G)$ by Rossi's theorem [4]. Consequently in Lemma 2. Let $\sigma_1, \sigma_2, \ldots, \sigma_p$ be mutually singular measures in $M_c(K)$, and let $z_1, z_2, \ldots, z_p$ be complex numbers satisfying $|z_j| \leqslant \|\sigma_j\|$ $(1 \leqslant j \leqslant p)$. We then claim that given $\nu \in L^1(\mu)$ there exists an element f in $\Delta_G \backslash \hat{G}$ such that

$$|f(\nu)| = r(\nu) \quad \text{and} \quad f(\sigma_j) = z_j \quad (1 \leqslant j \leqslant p). \tag{1}$$

There is no loss of generality in assuming $\|\sigma_j\| = 1$ for all j. Let τ_{2j-1} and τ_{2j} be mutually singular measures in $L^1(\sigma_j)$ such that $\sigma_j = (\tau_{2j-1} + \tau_{2j})/2$ and $\|\tau_{2j-1}\| = \|\tau_{2j}\| = 1$, and write $z_j = (w_{2j-1} + w_{2j})/2$ with $|w_{2j-1}| = |w_{2j}| = 1$. Since $m_G[C_0 + Gp(K)] = 0$, it follows from Lemma 3 that

$$\begin{aligned} &\left\| \left[\nu * \left(\delta_0 + \sum_{k=1}^{2p} \bar{w}_k \tau_k\right)\right]^n + M_a(G) \right\| \\ &\qquad = \left\| (\nu^n)_s * \left(\delta_0 + \sum_{k=1}^{2p} \bar{w}_k \tau_k\right)^n \right\| \\ &\qquad = \|(\nu^n)_s\| \left(1 + \sum_{k=1}^{2p} \|\tau_k\|\right)^n = \|(\nu^n)_s\|(1 + 2p)^n, \end{aligned} \tag{2}$$

which yields

$$r\left[\nu * \left(\delta_0 + \sum_{k=1}^{2p} \bar{w}_k \tau_k\right)\right] = r(\nu) \cdot (1 + 2p). \tag{3}$$

We can therefore find an element $f \in \Delta_G \backslash \hat{G}$ such that

$$|f(\nu)| = r(\nu) \quad \text{and} \quad f(\tau_k) = w_k \qquad (1 \leqslant k \leqslant 2p). \tag{1$'$}$$

By the choices of τ_k and w_k, (1)$'$ implies (1), which establishes our claim.

We next assert that, given $\nu \in L^1(\mu)$, every linear functional on $M_c(K)$, of norm $\leqslant 1$, extends to an element $f \in \Delta_G \backslash \hat{G}$ such that $|f(\nu)| = r(\nu)$. In fact, this is an easy consequence of (1) and the arguments of Hewitt and Kakutani in [6]. We leave the details to the reader.

Now choose three disjoint compact sets K_j in K $(j = 1, 2, 3)$, each of type M_0, and fix two probability measures $\lambda \in M_0(K_1)$ and $\tau \in M_0(K_2)$. We now prove that τ and the set

$$X = \{f \in \Delta_G: f(\lambda) = 1, |f(\lambda^*)| \leqslant 1/2\} \cup \{f \in \Delta_G: |1 - f(\lambda * \lambda^*)| \geqslant 3/2\}$$

have the required property. It is obvious that X is a compact subset of $\Delta_0 \backslash \Sigma_G$. Let c be a complex number of modulus $\leqslant 1$, and $\nu \in L^1(\mu)$. Let also φ be an arbitrary (linear) functional on $M_c(K_3)$, of norm $\leqslant 1$. By the Hahn-Banach theorem, φ extends to a functional ψ on $M_c(K)$, of norm one, such that $\psi(\lambda) = 1$ and $\psi(\tau) = c$. It follows from the result asserted in the last paragraph that there is an f in $\Delta_G \backslash \hat{G}$ such that $|f(\nu)| = r(\nu), f(\lambda) = 1, f(\tau) = c$ and $f = \varphi$ on $M_c(K_3)$. We want to show that such an f can be chosen from the set X. If $|f(\lambda^*)|$ is less than 1/2, then there is nothing to prove; so assume $|f(\lambda^*)| \geqslant 1/2$. Setting $\tau_1 = \lambda * \lambda^*$, we then have

$$\|(\nu^m)_s * \tau_1^n\| \geqslant |f(\nu^m * \tau_1^n)| \geqslant r(\nu)^m (1/2)^n$$

for all m and $n \in \mathbf{Z}^+$, so that

$$r[\nu^m * (\delta_0 - \tau_1)] \geqslant r(\nu)^m (3/2) \qquad (m \in \mathbf{Z}^+) \tag{4}$$

by (*) and Lemma 3. Putting $\mu_1 = \mu_0 * \exp(\tau_1)$, we also see that μ_1 is carried by the σ-compact set $C_1 = C_0 + Gp(K_1)$ and that

$$\mu_1[C_1 + Gp'(K_2 \cup K_3)] = \int \mu_0[C_1 + Gp'(K_2 \cup K_3) - y]\, d\theta(y)$$

$$\leqslant \mu_0[C_0 + Gp'(K)] \cdot e = 0,$$

where $\theta = \exp(\tau_1)$. Therefore, if $\tau_2, \ldots, \tau_p$ are mutually singular probability measures in $M_c(K_2 \cup K_3)$ and if $m, n, n_2, \ldots, n_p \in \mathbf{Z}^+$, then

$$\|[\nu^m * (\delta_0 - \tau_1)]^n * \tau_2^{n_2} * \cdots * \tau_p^{n_p} + M_a(G)\| \geqslant r[\nu^m * (\delta_0 - \tau_1)]^n \tag{5}$$

by Lemma 3 (applied to μ_1 and C_1). Consequently, one more application of Lemma 3, combined with (5), yields

$$r\left[\nu^m * (\delta_0 - \tau_1) * \left(\delta_0 + \sum_{j=2}^{p} \bar{z}_j \tau_j\right)\right] = r[\nu^m * (\delta_0 - \tau_1)] \cdot p \tag{6}$$

for all complex numbers $z_2, \ldots, z_p$ of absolute modulus one. (Notice that the left-hand side of (6) cannot be larger than the right-hand one.) Therefore there is a $g_m \in \Delta_G \backslash \hat{G}$ such that

$$|g_m[\nu^m * (\delta_0 - \tau_1)]| = r[\nu^m * (\delta_0 - \tau_1)], \quad g_m(\tau_j) = z_j \qquad (2 \leqslant j \leqslant p). \tag{7}$$

It follows from (4) and the first equality of (7) that $|1 - g_m(\tau_1)| \geqslant 3/2$, and so $g_m \in X$; moreover $|g_m(\nu)| = |g_m(\nu^m)|^{1/m} \geqslant r(\nu)\,(3/4)^{1/m}$ by (7) and (4). Recalling that X is compact and letting $m \longrightarrow \infty$, we find an element $h \in X$ such that

$$|h(\nu)| = r(\nu) \quad \text{and} \quad h(\tau_j) = z_j \qquad (2 \leqslant j \leqslant p). \tag{8}$$

We repeat almost the same argument as before to obtain an $f \in X$ with the required property. Since it is easy to prove that the conjugate space of $M_c(K_3)$ has cardinality equal to $2^{\mathfrak{c}}$, this establishes Theorem B for metrizable groups.

The proof for the nonmetrizable case is now easy. We first note that given $\nu \in M(G)$ there is a σ-compact open subgroup Γ of $\hat{G}$ such that $\|\nu * m_H\| = \|\nu\|$, there would exist a $\nu \in M(G)$ such that $\hat{\nu}(f) = 1$, $|\hat{\nu}| \leq 1$ on Δ_G, and $|\hat{\nu}| < 1/2$ on $\Delta_G \backslash \Sigma_G$. But then $r(\nu) = 1$, which contradicts Theorem B. This establishes part (b).

By Theorem B, no element of $\Delta_G \backslash \hat{G}$ can be a peak point for $M(G)$; hence $P_G \subset \hat{G}$. Therefore part (c) is an easy consequence of the fact that G is σ-compact if and only if $\hat{G}$ is metrizable [7]. This completes the proof.

PROOF OF COROLLARY 2. Let $\mu \in M(G)$ be given. Notice that $\Delta_G \backslash \hat{G}$ is the spectrum of the quotient algebra $M(G)/M_a(G)$. Choose a countable dense subset D of $\mathbf{C} \backslash \hat{\mu}(\Delta_G \backslash \hat{G})$. For each $c \in D$, there is a $\nu_c \in M(G)$ such that $\hat{\nu}_c = (c - \hat{\mu})^{-1}$ on $\Delta_G \backslash \hat{G}$. Setting $\Phi = \{\nu_c : c \in D\}$, we apply Theorem B to find a compact set X in $\Delta_0 \backslash \Sigma_G$ such that

$$\sup\{|c - \hat{\mu}(f)|^{-1} : f \in X\} = \sup\{|c - \hat{\mu}(g)|^{-1} : g \in \Delta_G \backslash \hat{G}\}$$

for all $c \in D$. Since $\hat{\mu}(X)$ is compact, this implies that $\hat{\mu}(X)$ contains all the boundary points of $\hat{\mu}(\Delta_G \backslash \hat{G})$ in $\mathbf{C}$.

If $\mathrm{Card}[\hat{\mu}(\Delta_0 \backslash \Sigma_G)] < \mathfrak{c}$, then $\hat{\mu}(\Delta_G \backslash \hat{G})$ has a countable boundary since it is compact. Therefore $\hat{\mu}(\Delta_G \backslash \hat{G})$ itself is countable, so that $\hat{\mu}(\Delta_G \backslash \hat{G}) = \hat{\mu}(\Delta_0 \backslash \Sigma_G)$ by the result already established. If $\hat{\mu}$ is real on $\Delta_0 \backslash \Sigma_G$, then $\hat{\mu}$ must be real on $\Delta_G \backslash \hat{G}$, hence $\hat{\mu}(\Delta_G \backslash \hat{G})$ has no interior point, and hence $\hat{\mu}(\Delta_G \backslash \hat{G}) = \hat{\mu}(\Delta_0 \backslash \Sigma_G)$. This establishes Corollary 2.

REMARKS. (a) Theorem A implies $\hat{\mu}_d(\Delta_G) \subset \hat{\mu}(\Delta_0 \backslash \Sigma_G)$ for all $\mu \in M(G)$. Moreover , we can prove that $\hat{\mu}_d(\Delta_G) \subset \hat{\mu}(\Delta_0 \cap \Sigma_G \backslash \hat{G})$ by applying the methods in [11].

(b) Notice that $\delta_0(C_1 + C_2) = 0$ if and only if $C_1 \cap (-C_2)$ is empty. If we only require that $C_0 \cap Gp'(K) = \varnothing$ in Lemma 2 instead of that $m_G[C_0 + Gp(K)] = \mu_0[C_0 + Gp'(K)] = 0$, then the assumption that C_0 is a σ-compact set with $m_G(C_0) = 0$ can be weakened to be that C_0 is a set of the first category in G (cf. [5, 2.1]).

(c) In some special cases, the proof of Theorem B can be somewhat simplified and we have a result slightly stronger than Theorem B.

Let H_0 be an open subgroup of G of the form $H_0 = \mathbf{R}^n \times H_1$, where n is a nonnegative integer and H_1 is a compact subgroup of G (cf. [10, 2.4.1]). Let P be the set of all $p \in \mathbf{Z}$ such that $1 \leq p < q(H_1)$ and $\mathrm{Card}\{\chi \in \hat{H}_1 : \chi^p = 1\} < \infty$. Then the last condition in Lemma 2 can be strengthened to be that $m_G[C_0 + Gp(K)] = \mu_0[C_0 + K(P)] = 0$. Here $K(P)$ denotes the set of all points x of the form $x = \Sigma_1^u k_j x_j$, where $u = u_x$ is a natural number, $x_1, x_2, \ldots, x_u$ are distinct elements of K, and $k_1, k_2, \ldots, k_u$ are integers such that $|k_j| \in P$ for some $1 \leq j \leq u$. The case where $2 \in P$ is particularly interesting.

Suppose in Lemma 3 that μ_0, C_0 and K are such that $\mu_0[C_0 + K(\{1, 2\})] = 0$. Then we can prove that

$$\|\nu * \sigma_1^{n_1} * \cdots * \sigma_p^{n_p}\| = \|\nu\| \cdot \|\sigma_1\|^{n_1} \cdots \|\sigma_p\|^{n_p}$$

for all $\nu \in L^1(\mu_0)$ and all $\sigma_j \in M_c(K_j \cup (-K_j))$. Therefore a moment's glance at the proof of Theorem B yields this result: If either $q(G) = 2$, or G contains an open subgroup H_0 as above with $2 \in P$, then the measure τ in Theorem B can be taken so that $\tau = \lambda * \lambda^*$ for some $\lambda \in M_0^+(G)$. We omit the details.

(d) If $\mu \in M^+(G)$, then the number $r(\mu)$ is in $\hat{\mu}(\Delta_0 \backslash \Sigma_G)$. To see this, choose a complex number z of absolute modulus one so that $zr(\mu) \in \hat{\mu}(\Delta_G \backslash \hat{G})$. Then we have

$$r(\delta_0 + \mu) = \lim_{n\to\infty} \|[(\delta_0 + \mu)^n]_s\|^{1/n}$$

$$\geqslant \lim_{n\to\infty} \|[(\delta_0 + \bar{z}\mu)^n]_s\|^{1/n} = 1 + r(\mu),$$

and so $r(\delta_0 + \mu) = 1 + r(\mu)$. Thus our assertion follows from Theorem B with $\Phi = \{\delta_0 + \mu\}$.

(e) Let $M_0^\infty(G)$ denote the L-ideal in $M(G)$ generated by all measures μ of the form $\mu = \mu_1 * \mu_2 * \cdots$, where the μ_j are probability measures in $M_0(G)$ and the infinite convolution product is assumed to converge in the weak-* topology of $M(G)$. Let also Δ_0^∞ denote the spectrum of $M_0(G)$ identified with an open subset of Δ_G. Then it is not difficult to prove that $\mathrm{Card}(\Delta_0 \backslash \Delta_0^\infty) \geqslant 2^{\mathfrak{c}}$. Moreover, in Theorem B, we can replace $M_0(G)$ and Δ_0 by $M_0^\infty(G)$ and Δ_0^∞, respectively. Using this result, we can prove that if Y is a boundary of $M_0^\infty(G)$, then $(Y \backslash \Sigma_G) \cup (Y \cap \hat{G})$ is a boundary of $M(G)$, which of course improves part (a) of Corollary 1. Similary the set Δ_0 in Corollary 2 can be replaced by Δ_0^∞.

(f) Finally we list three problems which the author has been unable to solve.

(i) Is it true that $\hat{\mu}(\Delta_G \backslash \hat{G}) = \hat{\mu}(\Delta_0 \backslash \Sigma_G)$ for all $\mu \in M(G)$?

(ii) Does $\hat{\mu}(\Sigma_G \backslash \hat{G}) = \{0\}$ imply $r(\mu) = 0$?

(iii) Does $\Sigma_G \backslash \hat{G}$ contain any strong boundary point for $M(G)$?

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